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An extension of the dynamics of one-dimensional wave splitting to three dimensions via Clifford algebra

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Abstract

The dynamics of split fields in one dimension are extended to three dimensions using Clifford algebra. The solutions of the resulting equations provide a unique insight into wave splitting and allow the construction of wave splittings in three dimensions that may be useful in solving the three-dimensional inverse scattering problem in the time domain.

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1. Introduction

Consider the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} - c^{-2}(x)\frac{\partial^2 u}{\partial t^2} = 0 \tag{1}$$

describing the propagation of scalar waves in a stratified medium characterized by a variable wave speed $c = c(x)$. Kristensson and Krueger [1] define a wave splitting for a function u satisfying equation (1) by

$$u^{\pm}(x,t) = \frac{1}{2}\left[u(x,t) \mp c(x)\int_{-\infty}^{t}\frac{\partial u}{\partial x}(x,s)\,\mathrm{d}s\right]. \tag{2}$$

It is easy to verify that if c is constant and

$$u(x,t) = f(x-ct) + g(x+ct)$$

is the general solution of (l), then

$$u^{+}(x,t) = f(x-ct) \qquad \text{and} \qquad u^{-}(x,t) = g(x+ct).$$

Moreover, this splitting defines a factorization of the wave equation (1) for constant c which is conveniently expressed in matrix notation by

$$\frac{1}{c}\frac{\partial}{\partial t}\begin{bmatrix} u^{+}(x,t) \\ u^{-}(x,t)\end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} u^{+}(x,t) \\ x^{-}(x,t)\end{bmatrix} = \begin{bmatrix} 0 \\ 0\end{bmatrix}. \tag{3}$$

If c is not constant in x, then $u^{\pm}$ as defined in equation (2) do not satisfy the wave equation. The wave splitting in this case provides a partial factorization of the wave equation in the sense that the first-order partial differential equations satisfied by u^{+} and u^{-} are coupled. These first-order equations can be written in matrix notation as

$$\frac{1}{c}\frac{\partial}{\partial t}\begin{bmatrix} u^{+}(x,t) \\ u^{-}(x,t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} u^{+}(x,t) \\ x^{-}(x,t) \end{bmatrix} = \frac{1}{2c}\frac{\mathrm{d}c}{\mathrm{d}x}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u^{+}(x,t) \\ x^{-}(x,t) \end{bmatrix} \qquad c = c(x) \tag{4}$$

known in the time-domain scattering literature as the dynamics. The dynamics provide much useful information about the directionality of wave propagation in the one-dimensional case; they can be used as the foundation for methods of study of scattering problems in layered media [2] by allowing the scattering system to be viewed as a system of inputs and outputs.

Much of the present work in time-domain scattering theory is devoted to extending these methods which work so well in one dimension to scattering problems where waves propagate in a medium characterized by a wave speed c which is a function of three spatial coordinates, say x_1, x_2 and x_3.

It is well-known in the physics literature [3] that Clifford algebras provide a method for the factorization of the wave equation in three dimensions. Dirac's equation for the electron was derived by factoring the Klein–Gordon equation using a matrix representation of a Clifford algebra. Of course, the Klein–Gordon equation reduces to the wave equation when the mass of the particle goes to zero, and Dirac's equation becomes the Weyl equations [4] for the neutrino in the same limit.

One recognizes in equations (3) and (4) one of the Pauli spin matrices [5]

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

which together with the identity matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

provide one of the simplest nontrivial examples of the matrix representation of the basis for a Clifford algebra [3]. The thought of using the remaining two Pauli spin matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}$$

to extend equation (4) to three dimensions is irresistible.

In this paper, it is shown that equation (4) can indeed be extended to three dimensions providing a partial factorization of the three-dimensional wave equation for nonhomogeneous media. The resulting equations, which can be written in the form

$$\left(I\frac{1}{c(\vec{r})}\frac{\partial}{\partial t} + \vec{\sigma}\cdot\vec{\nabla}\right)\underset{\sim}{\Psi}(\vec{r},t) = f(\underset{\sim}{\Psi}) \tag{5}$$

for Clifford algebra-valued field $\underset{\sim}{\Psi}$, where for $\vec{r} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$

$$\vec{\sigma}\cdot\vec{\Delta} = \sigma_1\frac{\partial}{\partial x_1} + \sigma_2\frac{\partial}{\partial x_2} + \sigma_3\frac{\partial}{\partial x_3}$$

are analysed. It is shown that by solving equation (5) one is simultaneously solving the three-dimensional wave equation and obtaining the direction of propagation, at least for singular waves.

2. The partial factorization of the three-dimensional wave equation

Rather than using a concrete representation such as the Pauli spin matrices, we let $\sigma_1, \sigma_2, \sigma_3$ and I be any square matrices having the following properties:

$$\sigma_i^2 = I \qquad \sigma_i\sigma_j + \sigma_j\sigma_i = 0 \qquad I^2 = I$$

and

$$\sigma_i I = I\sigma_i = \sigma_i$$

for all $i, j = 1, 2, 3, i \neq j$. The equations that result from this approach will be easier to interpret geometrically. It can be shown that the matrices

$$I, \sigma_1, \sigma_2, \sigma_3, \sigma_2\sigma_3, \sigma_3\sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_3$$

span an eight-dimensional, real vector space of matrices called a Clifford algebra [3]. Hence we can write any element $\underset{\sim}{\Psi}$ of this Clifford algebra in the form

$$\underset{\sim}{\Psi} = u_0 I + u_1\sigma_1 + u_2\sigma_2 + u_3\sigma_3 + u_{23}\sigma_2\sigma_3 + u_{31}\sigma_3\sigma_1 + u_{12}\sigma_1\sigma_2 + u_{123}\sigma_1\sigma_2\sigma_3.$$

If we let

$$u = u_0 \qquad v = u_{123} \qquad \vec{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k} \qquad \vec{v} = u_{23}\hat{i} + u_{31}\hat{j} + u_{12}\hat{k}$$

and

$$\vec{\sigma} = \sigma_1\hat{i} + \sigma_2\hat{j} + \sigma_3\hat{k}$$

then $\underset{\sim}{\Psi}$ can be written more compactly as

$$\underset{\sim}{\Psi} = uI + \vec{u}\cdot\vec{\sigma} + \sigma_1\sigma_2\sigma_3(vI + \vec{v}\cdot\vec{\sigma}).$$

We now suppose that $\underset{\sim}{\Psi}$ is a Clifford algebra solution of the equation

$$\left(I\frac{1}{c_0}\frac{\partial}{\partial t} + \vec{\sigma}\cdot\vec{\nabla}\right)\underset{\sim}{\Psi}(\vec{r}, t) = \underset{\sim}{0} \qquad c_0 = \text{constant}$$

where $\underset{\sim}{0}$ is the zero element of the Clifford algebra and $\vec{\nabla}$ is the usual gradient operator. Then it is easy to show that u, $\vec{u}$, v and $\vec{v}$ satisfy the equations

$$\vec{\nabla}\cdot\vec{u} = -\frac{1}{c_0}\frac{\partial u}{\partial t} \qquad \vec{\nabla}\cdot\vec{v} = -\frac{1}{c_0}\frac{\partial v}{\partial t}$$

$$-\frac{1}{c_0}\frac{\partial\vec{u}}{\partial t} + \vec{\nabla}\times\vec{v} = \vec{\nabla}u \qquad \frac{1}{c_0}\frac{\partial\vec{v}}{\partial t} + \vec{\nabla}\times\vec{u} = -\vec{\nabla}v.$$

We wish to extend these equations to the case of nonconstant c_0.

Let $c = c(\vec{r})$ be a function of position $\vec{r}$. We assume that u, $\vec{u}$, v and $\vec{v}$ satisfy the equations

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \Delta^2 u = 0 \qquad v = 0$$

$$-\frac{1}{c}\frac{\partial\vec{u}}{\partial t} + \vec{\nabla}\times\vec{v} = \vec{\nabla}u \qquad \frac{1}{c}\frac{\partial\vec{v}}{\partial t} + \vec{\nabla}\times\vec{u} = \vec{0}.$$

Then it is easy to show that u, $\vec{u}$ and $\vec{v}$ are related by the following equations if all fields go to zero as time t goes to minus infinity:

$$\vec{\nabla}\cdot\left(\frac{1}{c}\vec{u}\right) = -\frac{1}{c_2}\frac{\partial u}{\partial t} \qquad \text{and} \qquad \vec{\nabla}\cdot\left(\frac{1}{c}\vec{v}\right) = 0. \tag{6a}$$

Together with the equations

$$-\frac{1}{c}\frac{\partial \vec{u}}{\partial t} + \vec{\nabla}\times\vec{v} = \vec{\nabla}u \qquad \text{and} \qquad \frac{1}{c}\frac{\partial \vec{v}}{\partial t} + \vec{\nabla}\times\vec{u} = \vec{0} \tag{6b}$$

we have four equations bearing a remarkable similarity to the Maxwell equations. This being the case, we will on occasion call $\vec{u}$ the electric field and $\vec{v}$ the magnetic field corresponding to the solution of the wave equation u. Equations (6b) can be written as a symmetric hyperbolic system and consequently have a weak solution under suitable conditions on c [6].

Now it is straightforward to show that $\vec{\nabla}$ satisfies the equation

$$\left(I\frac{1}{c}\frac{\partial}{\partial t} + \vec{\sigma}\cdot\vec{\nabla}\right)\underset{\sim}{\Psi}(\vec{r},t) = \frac{\vec{\nabla}c}{c}\cdot(\vec{u}\vec{I} + \vec{v}\sigma_1\sigma_2\sigma_3) \tag{7}$$

corresponding to the extension of the partial factorization in equation (4) from the wave equation in one dimension to the three-dimensional case. By the way, the vectors $\vec{u}$ and $\vec{v}$ can be written in terms of $\underset{\sim}{\Psi}$ by using the Clifford algebra inner product [3] yielding

$$\left(I\frac{1}{c}\frac{\partial}{\partial t} + \vec{\sigma}\cdot\vec{\nabla}\right)\underset{\sim}{\Psi}(\vec{r},t) = \left\langle\vec{\sigma}\cdot\frac{\vec{\nabla}c}{c},\underset{\sim}{\Psi}\right\rangle I + \left\langle\sigma_1\sigma_2\sigma_3\vec{\sigma}\cdot\frac{\vec{\nabla}c}{c},\underset{\sim}{\Psi}\right\rangle\sigma_1\sigma_2\sigma_3$$

where $\langle\underset{\sim}{\Psi},\underset{\sim}{\Phi}\rangle$ is the coefficient of I in the basis expansion of the Clifford number $\underset{\sim}{\Psi}^+\underset{\sim}{\Phi}$ and $\underset{\sim}{\Psi}^+$ is the Clifford number reverse [3] of $\underset{\sim}{\Psi}$.

3. Recovery of the one-dimensional splitting

In this section, we show that equation (7) reduces to the familiar dynamics of one-dimensional wave propagation in a stratified medium. Let $\hat{n}$ be a constant unit vector, and let $\xi = \hat{n}\cdot\vec{r}$. We assume that $c = c(\xi)$ and $\underset{\sim}{\Psi} = \underset{\sim}{\Psi}(\xi,t)$. Then from the first of equations (6a) and (6b) it can be shown that

$$\frac{1}{c}\frac{\partial u}{\partial t} + \hat{n}\cdot\frac{\partial\vec{u}}{\partial\xi} = \frac{c'}{c}\hat{n}\cdot\vec{u} \tag{8a}$$

$$\frac{1}{c}\frac{\partial}{\partial t}\hat{n}\cdot\vec{u} + \frac{\partial u}{\partial\xi} = 0. \tag{8b}$$

From equation (8b) we see that

$$\hat{n}\cdot\vec{u} = -c\int_{-\infty}^{t} u\xi(\xi,s)\,\mathrm{d}s$$

using a subscript ξ to denote partial differentiation with respect to ξ.

Now let $u^+ = \frac{1}{2}(u + \hat{n}\cdot\vec{u})$ and $u^- = \frac{1}{2}(u - \hat{n}\cdot\vec{u})$. Then by adding and subtracting equations (8a) and (8b) one can show that

$$\frac{1}{c}\frac{\partial u^+}{\partial t} + \frac{\partial u^+}{\partial\xi} = \frac{c'}{2c}(u^+ - u^-)$$

$$\frac{1}{c}\frac{\partial u^-}{\partial t} - \frac{\partial u^-}{\partial\xi} = \frac{c'}{2c}(u^+ - u^-)$$

which are equivalent to equation (4) with ξ replacing x.

Another way to derive the one-dimensional dynamics is to choose a representation of $\sigma_1, \sigma_2, \sigma_3$ and I so that the direction of propagation $\hat{n}$ points in the x_3-direction and σ_3 is diagonal. Hence, assuming that $c = c(x_3)$ and $\underset{\sim}{\Psi} = \underset{\sim}{\Psi}(x_3, t)$, we choose

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $\sigma_1\sigma_2\sigma_3 = \mathrm{i}I$ and equation (7) reduces to

$$\left(I \frac{1}{c(x_3)} \frac{\partial}{\partial t} + \sigma_3 \frac{\partial}{\partial x_3} \right) \begin{bmatrix} u + u_3 + \mathrm{i}v_3 & u_1 + v_2 + \mathrm{i}(v_1 - u_2) \\ u_1 - v_2 + \mathrm{i}(v_1 + u_2) & u - u_3 - \mathrm{i}v_3 \end{bmatrix} = \frac{c'}{c}(u_3 + \mathrm{i}v_3)I.$$

Choosing the real part of this equation and noting that I and σ_3 are diagonal matrices we see that

$$\left(I \frac{1}{c} \frac{\partial}{\partial t} + \sigma_3 \frac{\partial}{\partial x_3} \right) \begin{bmatrix} u + u_3 \\ u - u_3 \end{bmatrix} = \frac{c'}{c} u_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We now let $u^+ = \frac{1}{2}(u+u_3)$ and $u^- = \frac{1}{2}(u-u_3)$. This again yields equation (4) with x replaced by x_3.

4. Solutions of the dynamical equations for homogeneous media

In an effort to gain insight into the meaning of the components of $\underset{\sim}{\Psi}$, we consider a special case. Consequently, suppose that the medium is homogeneous, so that c is a constant. Then equations (6) become

$$\vec{\nabla} \cdot \vec{u} = -\frac{1}{c}\frac{\partial u}{\partial t} \qquad \vec{\nabla} \cdot \vec{v} = 0$$

$$-\frac{1}{c}\frac{\partial \vec{u}}{\partial t} + \vec{\nabla} \times \vec{v} = \vec{\nabla} u \qquad \frac{1}{c}\frac{\partial \vec{v}}{\partial t} + \vec{\nabla} \times \vec{u} = \vec{0}.$$

It is now trivial to show that $\vec{v}$ satisfies the wave equation

$$\frac{1}{c^2}\frac{\partial^2 \vec{v}}{\partial t^2} - \nabla^2 \vec{v} = \vec{0}$$

implying that if $\vec{v} = \vec{0}$ everywhere as $t \to -\infty$ then $\vec{v} \equiv \vec{0}$ for all finite times. In this case, the electric field due to u is easily determined to be

$$\vec{u}(\vec{r}, t) = -c\vec{\nabla} \int_{-\infty}^{t} u(\vec{r}, s)\, \mathrm{d}s$$

which is a solution of the wave equation.

If u is a plane wave propagating in the direction $\hat{n}$ we can write

$$u = f(\hat{n} \cdot \vec{r} - ct)$$

for some function f of compact support in $\mathbb{R}$. Then

$$\vec{u} = \hat{n}u \qquad \text{and} \qquad u = \hat{n} \cdot \vec{u}$$

and we note that $\vec{u}$ is parallel to the direction of propagation of u. Now suppose that u is a spherical wave propagating outward (upper sign) or inward (lower sign) from the origin, i.e.

$$u = \frac{f(r \mp ct)}{r}.$$

Then, if $\hat{r}$ is a unit vector in the radial direction away from the origin

$$\vec{u} = \pm\hat{r}\left(u \pm \frac{c}{r}\int_{-\infty}^{t} u\,\mathrm{d}s\right)$$

which is a solution of the wave equation. However,

$$\pm\hat{r}\cdot\vec{u} = u \pm \frac{c}{r}\int_{-\infty}^{+} u\,\mathrm{d}s$$

is not a solution. Consequently, the splitting defined by $u^{\pm} = \frac{1}{2}[u \pm \hat{r}\cdot\vec{u}]$ is only approximate.

In both the cases, we see that the vector $\vec{u}$ yields the direction of propagation of the wave u. However, if one takes the scalar product of $\vec{u}$ with the direction of propagation in order to construct a splitting along the lines suggested by section 3, one arrives at a solution of the wave equation only in the plane wave case. The difficulty is seen to lie in the fact that the wave operator

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

commutes with the gradient operator $\vec{\nabla}$ but not with the components of the gradient, except in rectangular coordinates. Nevertheless, if the gradient vector is expressed in curvilinear coordinates, then this yields a fruitful method of producing splittings that involve only local operators [7] but that are approximate even in homogeneous media.

5. The magnetic field $\vec{v}$

We now determine the significance of the magnetic field $\vec{v}$. As we saw in the previous section, if the medium is homogeneous, then $\vec{v}$ can be set identically to zero for all time. Since closed-form solutions of the wave equation in inhomogeneous media are limited in number, we must appeal to asymptotic analysis. We will consider two approaches, a perturbation analysis for a slowly varying medium and a propagation of singularities argument. Undoubtedly, other forms of analysis will yield useful information.

Before engaging in this asymptotic analysis, we prove that if $\vec{\nabla}c \times \vec{\nabla}u \neq \vec{0}$ in a medium of nonconstant c, then $\vec{v} \not\equiv \vec{0}$. Consider equations (6*a*) and (6*b*) and assume that $\vec{v} \equiv \vec{0}$. Then u and $\vec{u}$ satisfy the equations

$$\vec{\nabla}\cdot\left(\frac{1}{c}\vec{u}\right) = -\frac{1}{c^2}\frac{\partial u}{\partial t} \qquad -\frac{\partial}{\partial t}\left(\frac{1}{c}\vec{u}\right) = \vec{\nabla}u \qquad \text{and} \qquad \vec{\nabla}\times\vec{u} = \vec{0}.$$

Integrating the second equation yields

$$\vec{u} = -c\vec{\nabla}\int_{-\infty}^{t} u(\vec{r}, s)\,\mathrm{d}s$$

which also satisfies the first equation, since u is a solution of the wave equation. However, substituting into the third equation, we see that

$$\vec{0} = \vec{\nabla}\times\vec{u} = -\vec{\nabla}\times\left(c\vec{\nabla}\int_{-\infty}^{t} u\,\mathrm{d}s\right) = -\vec{\nabla}c\times\vec{\nabla}\int_{-\infty}^{t} u\,\mathrm{d}s$$
$$= -\int_{-\infty}^{t}\vec{\nabla}c\times\vec{\nabla}u\,\mathrm{d}s.$$

Hence, differentiating with respect to t

$$\vec{\nabla} c \times \vec{\nabla} u = \vec{0}$$

which concludes the proof. We see that, in a way analogous to gauge fields, if the medium loses symmetry, then a new field must be introduced.

An asymptotic analysis allows us to determine a quantitative connection between $\vec{\nabla} c \times \vec{\nabla} u$ and $\vec{v}$. First we consider a slowly varying medium [8]. We assume that the speed of propagation c is given by a power series expansion in a small parameter ε:

$$c = c(\varepsilon \vec{r}) = c_0 + \varepsilon \vec{c}_1 \cdot \vec{r} + 0(\varepsilon^2)$$

where $\vec{c}_1$ is a constant vector and $0(\varepsilon^2)$ represents terms of order ε^2 and higher.

We now suppose that u, $\vec{u}$ and $\vec{v}$ have analogous power series expansions in ε:

$$u = u_0 + \varepsilon u_1 + 0(\varepsilon^2)$$
$$\vec{u} = \vec{u}_0 + \varepsilon \vec{u}_1 + 0(\varepsilon^2)$$
$$\vec{v} = \vec{v}_0 + \varepsilon \vec{v}_1 + 0(\varepsilon^2).$$

Substitution of these expansions as well as the above expansion of c into equations ($6a$) and ($6b$) and equating coefficients of like powers of ε yields

$$\frac{1}{c_0^2}\frac{\partial^2 u_0}{\partial t^2} - \nabla^2 u_0 = 0$$
$$\vec{u}_0 = -c_0 \vec{\nabla} \int_{-\infty}^{t} u_0(\vec{r}, s)\,\mathrm{d}s$$
$$\vec{v}_0 = \vec{0}$$

and

$$\frac{1}{c_0^2}\frac{\partial^2 \vec{v}_1}{\partial t^2} - \nabla^2 \vec{v}_1 = \frac{\vec{c}_1}{c_0} \times \vec{\nabla} u_0.$$

The first equation for u_0 is to be expected since u is a solution of the wave equation. Considerations similar to those of the previous section show that the electric field $\vec{u}$ parallels the direction of propagation of u, at least to lowest order in ε. From the third and the fourth equations, we can conclude that the magnetic field $\vec{v}$ is of first order in ε and satisfies a wave equation with $\vec{\nabla} c \times \vec{\nabla} u_0$ as its source, to lowest order in ε, and is consistent with our earlier result.

More revealing conclusions can be drawn by pursuing a propagation of singularity analysis. We first recall familiar results about the wave equation [8]

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = 0.$$

Expand u as follows:

$$u = \sum_{j=0}^{\infty} u_j H_j[\phi]$$

where

$$H_j[x] = \frac{x^j}{j!} H(x)$$

and H is the Heaviside function. Substitution of this expansion for u into the wave equation yields the eikonal equation

$$\phi_t^2 - c^2|\vec{\nabla}\phi|^2 = 0$$

and the transport equations

$$2\frac{\partial u_0}{\partial t}\phi_t - 2c^2\vec{\nabla}u_0 \cdot \vec{\nabla}\phi + u_0(\phi_{tt} - c^2\nabla^2\phi) = 0$$

$$2\frac{\partial u_j}{\partial t}\phi_t - 2c^2\vec{\nabla}u_j \cdot \vec{\nabla}\phi + u_j(\phi_{tt} - c^2\nabla^2\phi) = -\frac{\partial^2 u_{j-1}}{\partial t^2} + c^2\nabla^2 u_{j-1} \qquad j \geqslant 1.$$

Equations (6b) can be treated in the same way. We expand the vector fields $\vec{u}$ and $\vec{v}$ as

$$\vec{u} = \sum_{j=0}^{\infty} \vec{u}_j H_j[\phi] \qquad \text{and} \qquad \vec{v} = \sum_{j=0}^{\infty} \vec{v}_j H_j[\phi].$$

Substitution into (6b) yields the eikonal equation for ϕ. It is convenient to write ϕ as

$$\phi(x, y, z, t) = \psi(x, y, z) - t$$

so that all fields are progressive waves. Then the eikonal equation becomes

$$c^2|\vec{\nabla}\psi|^2 = 1.$$

Moreover, we generate the transport equations

$$\vec{\nabla}\psi \times \vec{u}_0 - \frac{1}{c}\vec{v}_0 = \vec{0}$$

$$\vec{\nabla}\psi \times \vec{v}_0 + \frac{1}{c}\vec{u}_0 = u_0\vec{\nabla}\psi$$

$$\vec{\nabla}\psi \times \vec{u}_j - \frac{1}{c}\vec{v}_j = -\frac{1}{c}\frac{\partial v_{j-1}}{\partial t} - \vec{\nabla} \times \vec{u}_{j-1} \qquad j \geqslant 1$$

and

$$\vec{\nabla}\psi \times \vec{v}_j + \frac{1}{c}\vec{u}_j = \frac{1}{c}\frac{\partial \vec{u}_{j-1}}{\partial t} - \vec{\nabla} \times \vec{v}_{j-1} + u_j\vec{\nabla}\psi - \vec{\nabla}u_{j-1} \qquad j \geqslant 1.$$

Considering only the most singular terms $\vec{u}_0$ and $\vec{v}_0$, we see from the first of these equations that the ray direction $c\vec{\nabla}\psi$ is perpendicular to the magnetic field $\vec{v}_0$, and the angle between the ray direction and $\vec{u}_0$ is easily calculated from the second equation. Hence, it is reasonable to conclude that in solving equation (7) one is computing essential asymptotic information about the direction of propagation of the wave field.

6. Conclusions

We have seen that it is possible to extend the dynamical equations of one-dimensional time-domain scattering theory to three dimensions by use of Clifford algebra. The resulting equations provide a unique insight into the dynamics of split fields since it appears that in addition to an 'electric' field appearing in the one-dimensional case as the time integral of the spatial derivative of the wave field, a further 'magnetic' field must be introduced in the case of nonhomogeneous media, that is, when the symmetry of the homogeneous medium is destroyed. However, the computation of both vector fields allows one to deduce the direction of propagation of the wave field as in the one-dimensional case. Whether these observations can be used to step through a medium systematically and solve the inverse problem remains to be proved.

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